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DUAL QUATERNION CLOSED FORM EQUATIONS OF SPATIAL $7R$ LOOPS

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ABSTRACT. By using the double quaternions, 16 finite-form algebraic equations are derived that express the conditions for the closure of the $7R$ mechanism. It was possible to significantly simplify these equations by introducing intermediate angles. The unknowns are sines and cosines of intermediate angles, which are linear combinations of output angles of rotation in kinematic pairs. The first six equations of the 16 are linear with respect to the unknowns, the following 8 equations express the equality to unity of the sum of squares of sines and cosines, and the remaining 2 equations express additional connections between the unknowns. Numerical results are obtained.

Keywords: quaternions, dual numbers, manipulator, space transformations, kinematic analysis.

AMS Subject Classification: 70B15, 70E15, 70M20.

1. Introduction

It is well-known that the displacement analysis of the general $7R$ mechanism is the most difficult task in the analysis of single-loop spatial mechanism. It is "The Mount Everest of Kinematic Problems" - Freudenstein [10]. The main difficulty in solving this problem lies, firstly, in obtaining algebraic equations, connecting variable angles in rotational kinematic pairs, and secondly in solving these equations. Therefore, the successful solution of the problem mainly depends on the complexity of the equations obtained. In order to get the closed form analysis of this problem, many mathematical methods have been used, such as recursive notation [20], matrices with real-number elements [5]. The work of a number of researchers is devoted to the study of spatial mechanisms using quaternion algebra, dual numbers and screw theory. In the works of F.H. Mammadov [17, 16], dual quaternions are used to compile closed-loop equations. By introducing intermediate unknown angles, the author will greatly simplify these equations. Using the principle of transferring Kotelnikov, the problem of speeds and accelerations of spatial mechanisms is also solved. In the work of F.M. Dimentberg [6], the theory of screws, the algebra of dual numbers, the kinematic analysis of spatial mechanisms based on screw calculus are described, various groups of screws are described. V.N. Branets and I.P. Shmyglevsky [1] describe in detail the algebra of quaternions, their properties, and the possibility of using as a unified operator of spatial orientation of a solid body. The authors conducted an extensive study of the kinematics of a solid in quaternion representation. In [12] D. Gan et al. describe the algebra of the dual quaternion and its use in the mechanisms of sequential structure. The solution of the problem of displacement of a single-circuit spatial $7R$ mechanism is considered. To solve closed-loop equations, Dixon resultant is used. The results of the numerical example are given, including four real assemblies of the $7R$ mechanism identified. In [8] J. Duffy and S.

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Derby, a 24^{th} degree equation was obtained between the input and output angles for a spatial $7R$ mechanism with successively perpendicularly intersecting axes. Solving the problem of moving this mechanism is essentially an important step towards solving the $7R$ mechanism of the most general form with rotational kinematic pairs arbitrarily oriented in space.

In [7], J. Duffy and C. Crane expressed the relationship between the input and output angles of the spatial $7R$ mechanism with the axes of rotational kinematic pairs arbitrarily located in space, in the form of the zero determinant 16×16, whose elements are expressed by the tangents of half angles of rotation in rotational kinematic pairs. An algorithm has been developed for calculating the values of angular displacements in the remaining rotational pairs of the mechanism. The results are confirmed by numerical examples. J. Duffy in his article [9] obtained the eighth degree equations between the input and output displacement for the spatial seven link mechanisms *RPPRRRR, RRRPPRR, RPRRRPR, RPRRPRR* and *RPRPRRR*. The results are confirmed by numerical examples. Solving the problem of moving these mechanisms is a significant advance in the kinematic analysis of spatial mechanisms. In addition, the article discusses the definition of the displacements of the spatial seven-link *4R-3P*, 5-link *3R-2C* and 6-link *4R-PC* mechanisms. Chen Wei-rong in article [3] derived the equation of the relationship between the input and output displacements of the spatial $7R$ mechanism. To do this, he uses the rotation matrices as the operator of spatial orientation of solids; he obtains the scalar products of the direction cosines of the unit vectors of the Cartesian coordinate system and derives 6 equations of the closure condition. George N. Sandor et al. in article [20] present conditions for excluding cases of branching during the synthesis of the spatial $7R$ mechanism. These conditions are based on the theory of Hunt on linear-dependent screws and the theory of the existence of a stationary configuration. The paper presents a numerical example of the synthesis of the $7R$ mechanism with the exception of the cases of branching. Hong YouLee and Chong Gao Liang [13] obtained a 16^{th} degree equation expressing the relationship between the tangents of the half angles of rotation of the input and output link of the $7R$ spatial mechanism. The input-output equation is obtained from the condition that the 8×8 determinant is zero, and the relationship between the other angles is expressed by the equation in an implicit form. The authors conducted a detailed analysis based on vector analysis and the algebra of dual numbers, developed by J. Duffy. The results are confirmed by numerical examples. Martin Pfurner et al. in [19] analyzes a seven-link single-loop mechanism with one degree of freedom of a variable structure by combining two four-link spatial mechanisms with redundant coupling with rotational and translational pairs, namely the Bennett mechanism with a spatial *RPRP* mechanism with excess coupling. Both initial mechanisms are connected in such a way that one parameter of movement and two points of initial mechanisms coincide. Subsequently, all unnecessary links are removed, and the remaining seven links, joined at these points, are inserted to create a single-loop mechanism. The mechanisms of $7R$, *5R2P*, and *4R3P* are created in this way. The kinematic analysis of the *7*-link mechanisms of the variable structure of all classes is considered in detail. Alfredo Valverde and Panagiotis Tsiotras [21] describe double quaternions, the basic mathematical operations on them, the kinematic study of robotic devices using double quaternions, as well as the elimination of various motion restrictions arising in the course of spacecraft flight. In [11] Jaime Gallardo-Alvarado et al. the displacement analysis of the spherical parallel manipulator is simplified by formulating closure equations based on two unit vectors representing the orientation of the moving platform and solved the problems of displacement. After, the input-output equation for speed is obtained using the theory of screws. Chelnokov [2], on the basis of the Kotelnikov-Study principle of transfer, presented the spatial motion of a rigid body, which is a combination of translational and rotational motions in one helical motion. Possible applications in the theory of spatial mechanisms and mechanics of robotic manipulators, as well as in inertial navigation problems for determining the orientation and speed of a moving object are indicated. In [14], Jing Li combined a quaternion-based rotation vector and a double quaternion-based screw vector. First, the rotation vector method is used to update the attitude quaternion of the leader rigid body relative to the inertial coordinate system. Then, the screw vector algorithm is used to update the dual quaternion of the follower rigid body relative to the inertial coordinate system. Finally, the relative position and relative attitude updating algorithms of the leader-follower rigid body are established based on the dual quaternion.

Neil T. Dantam in paper [4] has presented a new derivation of the dual quaternion in exponential and logarithmic form, eliminating the singularity. It is shown that the implicit representation of dual quaternions provides analytical and numerical advantages over matrices and ordinary dual quaternions, and they are more compact and require fewer arithmetic operations. In the article [15] Ping Feng Lin et al. introduced a new approach to analytical solution of the inverse problem of kinematics. The author claims that within the framework of the theory of screws, this process is mathematically performed much more efficiently and has a geometric meaning. Both the results obtained and the simulation results are compared with the general the numerical solutions used and analytical solutions based on matrices show that the proposed method is faster and is not subject to numerical instability caused by the proximity to a singular configuration. P.S. Pankov et al. in [18] show that many results on the asymptotic behavior of solutions of dynamical systems can be uniformly formulated using the new concepts of "asymptotic equivalence" and "asymptotic reduction of the dimension of the space of solutions". The new method of space splitting makes it possible to extend the phenomenon of singular solutions to large classes of operator-difference equations and to obtain new results for differential equations with delay.

In this paper, as the operator of the most general spatial transformation, it is proposed to use dual quaternions, which are used to compose the closed-loop equations of closure of the $7R$ spatial mechanism of the most general form, and lay down the conditions for simplifying these equations. The obtained equations are applied to the numerical solution of the problem of displacements of the $7R$ mechanism.

2. A brief note on quaternions, the operation of turning and dual numbers

A quaternion is a complex number composed of a real unit 1 and three imaginary units $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$ with real elements:

$$\boldsymbol{\lambda} = 1\lambda_0 + \sum_{k=1}^{3} \lambda_k \bar{i}_k. \tag{1}$$

The rules for multiplying units are as follows:

$$1 \circ \bar{\mathrm{i}}_1 = \bar{\mathrm{i}}_1 \circ 1 = \bar{\mathrm{i}}_1,\ 1 \circ \bar{\mathrm{i}}_2 = \bar{\mathrm{i}}_2 \circ 1 = \bar{\mathrm{i}}_2,\ 1 \circ \bar{\mathrm{i}}_3 = \bar{\mathrm{i}}_3 \circ 1 = \bar{\mathrm{i}}_3,\ 1 \circ 1 = 1,$$
$$\bar{\mathrm{i}}_1 \circ \bar{\mathrm{i}}_1 = -1,\ \bar{\mathrm{i}}_2 \circ \bar{\mathrm{i}}_2 = -1,\ \bar{\mathrm{i}}_3 \circ \bar{\mathrm{i}}_3 = -1,$$
$$\bar{\mathrm{i}}_1 \circ \bar{\mathrm{i}}_2 = -\bar{\mathrm{i}}_2 \circ \bar{\mathrm{i}}_1 = \bar{\mathrm{i}}_3,\ \bar{\mathrm{i}}_3 \circ \bar{\mathrm{i}}_1 = -\bar{\mathrm{i}}_1 \circ \bar{\mathrm{i}}_3 = \bar{\mathrm{i}}_2,\ \bar{\mathrm{i}}_2 \circ \bar{\mathrm{i}}_3 = -\bar{\mathrm{i}}_3 \circ \bar{\mathrm{i}}_2 = \bar{\mathrm{i}}_1.$$

The rules for multiplying imaginary units are remembered using the following principle: when multiplying two units located along the arrow, the third unit with the sign "+" is obtained; when moving in the opposite direction; the unit is obtained with the sign "-". Results multiplication of real and imaginary units were shown in Table 1. These rules indicate that multiplication by 1 does not change the quaternion; therefore in the future the first term λ_0 will be denoted without unity in the quaternion expression.

Table 1. Units' multiplication.

	1	$\bar{i}_1$	$\bar{i}_2$	$\bar{i}_3$
1	1	$\bar{i}_1$	$\bar{i}_2$	$\bar{i}_3$
$\bar{i}_1$	$\bar{i}_1$	-1	$\bar{i}_3$	$-\bar{i}_2$
$\bar{i}_2$	$\bar{i}_2$	$-\bar{i}_3$	- 1	$\bar{i}_1$
$\bar{i}_3$	$\bar{i}_3$	$\bar{i}_2$	$-\bar{i}_1$	-1

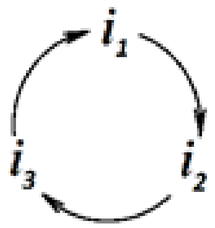

Figure 1. The rules for multiplying imaginary units.

The units $\bar{i}_1, \bar{i}_2, \bar{i}_3$ can be identified with the units of the three-dimensional space and consider the coefficients at these units as components of the vector. Accordingly, the quaternion can be represented as a sum of scalar and vector parts:

$$\boldsymbol{\lambda}{=}1\lambda_0 + \lambda_1\bar{i}_1 + \lambda_2\bar{i}_2 + \lambda_3\bar{i}_3 = sqal\boldsymbol{\lambda} + vect\boldsymbol{\lambda},$$

where $sqal\boldsymbol{\lambda}{=}\lambda_0$, $vect\boldsymbol{\lambda} = \lambda_1\bar{i}_1 + \lambda_2\bar{i}_2 + \lambda_3\bar{i}_3$.

A conjugate is called a quaternion, in which the vector part has the opposite sign:

$$\widetilde{\boldsymbol{\lambda}}{=}sqal\boldsymbol{\lambda} - vect\boldsymbol{\lambda} = \lambda_0 - \lambda_1\bar{i}_1 - \lambda_2\bar{i}_2 - \lambda_3\bar{i}_3.$$

Multiplication of quaternions has associative and distributive properties:

$$(\boldsymbol{\lambda}_1\boldsymbol{\lambda}_2)\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_1(\boldsymbol{\lambda}_2\boldsymbol{\lambda}_3), \quad \boldsymbol{\lambda}_1(\boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_3) = \boldsymbol{\lambda}_1\boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_1\boldsymbol{\lambda}_3.$$

But the multiplication of quaternions is not commutative. Indeed, after performing quaternion multiplication of two quaternions λ and μ, we obtain:

$$\begin{aligned} \boldsymbol{\lambda} \circ \boldsymbol{\mu} = \lambda_0\mu_0 - \lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 + \lambda_0\left(\mu_1\bar{i}_1 + \mu_2\bar{i}_2 + \mu_3\bar{i}_3\right) + \\ +\mu_0\left(\lambda_1\bar{i}_1 + \lambda_2\bar{i}_2 + \lambda_3\bar{i}_3\right) + \begin{vmatrix} \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \end{aligned} \tag{2}$$

It follows from the expression obtained that $\boldsymbol{\lambda} \circ \boldsymbol{\mu} = \boldsymbol{\mu} \circ \boldsymbol{\lambda}$ only when the determinant disappears. The latter is possible either when $\lambda_1 = \lambda_2 = \lambda_3 = 0$, or $\mu_1 = \mu_2 = \mu_3 = 0$, that is, when one of the factors is a scalar, or when $\boldsymbol{\lambda} = a\boldsymbol{\mu}$ (a is a real number). From the last expression, we also conclude that the quaternion multiplication of two vectors contains the scalar and vector products of these vectors. Indeed, if we take in the formula (2) $\lambda_0 = \mu_0 = 0$, then we get:

$$\boldsymbol{\lambda} \circ \boldsymbol{\mu} = -\lambda_1\mu_1 - \lambda_2\mu_2 - \lambda_3\mu_3 + \begin{vmatrix} \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}.$$

The norm of a quaternion is the product of $\boldsymbol{\lambda}$ by the conjugate quaternion $\widetilde{\boldsymbol{\lambda}}$:

$$\lambda = \boldsymbol{\lambda} \circ \widetilde{\boldsymbol{\lambda}} = \widetilde{\boldsymbol{\lambda}} \circ \boldsymbol{\lambda} = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

This expression is obtained on the basis of expression (2). The quaternion norm is denoted by $|\boldsymbol{\lambda}|$ or λ. If $|\boldsymbol{\lambda}| = 1$, the quaternion is called unit quaternion. In the future, we will use only unit quaternions.

Any quaternion (1) can be represented in a trigonometric form:

$$\boldsymbol{\lambda} = \lambda(\cos\varphi + \mathbf{e}\sin\varphi).$$

Where λ is the quaternion norm, $\mathbf{e}$ is the unit vector of the vector part of the quaternion λ:

$$\mathbf{e} = \frac{vect\lambda}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}} = \frac{\lambda_1\bar{\mathbf{i}}_1 + \lambda_2\bar{\mathbf{i}}_2 + \lambda_3\bar{\mathbf{i}}_3}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}, \tag{3}$$

$$\cos\varphi = \frac{\lambda_0}{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}, \quad \sin\varphi = \frac{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}{\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2}. \tag{4}$$

Accordingly, the trigonometric expression for the unit quaternion will have the following form:

$$\boldsymbol{\lambda} = \cos\varphi + \mathbf{e}\sin\varphi. \tag{5}$$

The quaternion algebra allows us to represent the spatial transformation in a simple form, the essence of which is as follows. Let λ and $\mathbf{r}$ are non-scalar quaternions, then the quaternion multiplication

$$\mathbf{r}' = \boldsymbol{\lambda} \circ \mathbf{r} \circ \widetilde{\boldsymbol{\lambda}}. \tag{6}$$

There is also a quaternion, whose norm and scalar part are equal to the norm and the scalar part of the quaternion $\mathbf{r}$. The vector part of $\mathbf{r}'$ *vect* $\mathbf{r}'$ is obtained by rotating *vect*$\mathbf{r}$ along the cone about the $\mathbf{e}$ axis by a double angle 2φ. Operation (6) only changes the vector part of the quaternion, so this operation can be considered as an operation of transforming the vector $\mathbf{r}$ into a vector $\mathbf{r}'$:

$$\begin{aligned}
r_0' &= r_0\\
r_1' &= \frac{2\left(\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2\right)}{\lambda^2} r_1 + \frac{2\left(\lambda_1\lambda_2-\lambda_0\lambda_3\right)}{\lambda^2} r_2 + \frac{2\left(\lambda_1\lambda_3+\lambda_0\lambda_2\right)}{\lambda^2} r_3,\\
r_2' &= \frac{2\left(\lambda_1\lambda_2+\lambda_0\lambda_3\right)}{\lambda^2} r_1 + \frac{\lambda_0^2+\lambda_2^2-\lambda_1^2-\lambda_3^2)}{\lambda^2} r_2 + \frac{2\left(\lambda_2\lambda_3-\lambda_0\lambda_1\right)}{\lambda^2} r_3,\\
r_3' &= \frac{2\left(\lambda_1\lambda_3-\lambda_0\lambda_2\right)}{\lambda^2} r_1 + \frac{2\left(\lambda_2\lambda_3+\lambda_0\lambda_1\right)}{\lambda^2} r_2 + \frac{\lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2)}{\lambda^2} r_3.
\end{aligned} \tag{7}$$

If quaternion λ, which specifies the transformation (6) is unit, then expression (7) takes the following more simpler form:

$$\begin{aligned}
r_1' &= \left(\lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2\right) r_1 + 2\left(\lambda_1\lambda_2-\lambda_0\lambda_3\right) r_2 + 2\left(\lambda_1\lambda_3+\lambda_0\lambda_2\right) r_3,\\
r_2' &= 2\left(\lambda_1\lambda_2+\lambda_0\lambda_3\right) r_1 + (\lambda_0^2+\lambda_2^2-\lambda_1^2-\lambda_3^2) r_2 + 2\left(\lambda_2\lambda_3-\lambda_0\lambda_1\right) r_3,\\
r_3' &= 2\left(\lambda_1\lambda_3-\lambda_0\lambda_2\right) r_1 + 2\left(\lambda_2\lambda_3+\lambda_0\lambda_1\right) r_2 + (\lambda_0^2+\lambda_3^2-\lambda_1^2-\lambda_2^2) r_3.
\end{aligned} \tag{8}$$

Suppose further that the vector $\mathbf{r}$ is subjected to successive transformations-rotations given by the quaternions $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_n$, respectively. The resulting rotation is determined by the quaternion $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_n \circ \boldsymbol{\lambda}_{n-1} \circ \cdots \circ \boldsymbol{\lambda}_1, \tag{9}$$

where the quaternions $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \cdots, \boldsymbol{\lambda}_n$ are expressed in the original coordinate system. Of course, with the increase in the number of successive transformations, the use of the latter expression becomes time-consuming. But if we use the Rodrigues-Hamilton parameters as the quaternions of sequential rotations, then the resulting quaternion is determined by the following expression [1]:

$$\boldsymbol{\lambda} = \boldsymbol{\lambda_1} \circ \boldsymbol{\lambda_2} \circ \cdots \circ \boldsymbol{\lambda_n}. \tag{10}$$

The components of the quaternion in a basis that is transformed by the same quaternion are called the Rodrigues-Hamilton parameters. This quaternion has equal components in both coordinate systems due to the fact that it is quaternion that determines by transformation from one coordinate system to another.

The dual number has the following form:

$$A = a + \delta a^0,$$

where a is the main part of the dual number, a^0 is the moment part of the dual number, δ is the Clifford operator with the property $\delta^2 = 0$. The dual numbers are denoted by large letters. The basic operations on the dual numbers are carried out according to the formulas:

$$A \pm B = (a \pm b) + \delta \left(a^0 \pm b^0\right),$$

$$AB = ab + \delta \left(a^0 b + ab^0\right),$$

$$\frac{A}{B} = \frac{a}{b} + \delta \frac{a^0 b + ab^0}{b^2}, \quad (b \neq 0).$$

The dual function has the following form:

$$F(X) = f\left(x + \delta x^0\right) = f(x) + \delta x^0 f'(x)$$

$$F(X,\ A_1,\ A_2,\ \ldots,\ A_n) =$$
$$F(x,\ a_1,\ a_2,\ \cdots,\ a_n) + \delta\left(x^0\frac{dF}{dx} + a_1^0\frac{dF}{da_1} + a_2^0\frac{dF}{da_2} + \cdots + a_n^0\frac{dF}{da_n}\right).$$

The trigonometric functions of the dual number $X = x + \delta x^0$ are expressed as follows:

$$\sin X = \sin x + \delta x^0 \cos x, \quad \cos X = \cos x - \delta x^0 \sin x, \quad \tan X = \tan x + \delta x^0 \frac{1}{\cos^2 x}.$$

If in the expression (1) the real numbers $\lambda_o, \lambda_1, \lambda_2, \lambda_3$ are replaced by the dual ones, we obtain the expression for the dual quaternion:

$$\boldsymbol{\Lambda} = \Lambda_o + \Lambda_1 \bar{\mathrm{i}}_1 + \Lambda_2 \bar{\mathrm{i}}_2 + \Lambda_3 \bar{\mathrm{i}}_3, \tag{11}$$

where $\Lambda = \lambda + \delta\lambda^0$, $(k = 0, 1, 2, 3)$ components of the dual quaternion. We transform expression (11):

$$\begin{aligned} &\boldsymbol{\Lambda} = \Lambda_0 + \Lambda_1 \bar{\mathrm{i}}_1 + \Lambda_2 \bar{\mathrm{i}}_2 + \Lambda_3 \bar{\mathrm{i}}_3 = \\ &= \left(\lambda_0 + \delta\lambda_0^0\right) + \left(\lambda_1 + \delta\lambda_1^0\right)\bar{\mathrm{i}}_1 + \left(\lambda_2 + \delta\lambda_2^0\right)\bar{\mathrm{i}}_2 + (\lambda_3 + \delta\lambda_3^0)\bar{\mathrm{i}}_3 = \\ &= \lambda_0 + \lambda_1 \bar{\mathrm{i}}_1 + \lambda_2 \bar{\mathrm{i}}_2 + \lambda_3 \bar{\mathrm{i}}_3 + = \delta\left(\lambda_0^0 + \lambda_1^0 \bar{\mathrm{i}}_1 + \lambda_2^0 \bar{\mathrm{i}}_2 + \lambda_3^0 \bar{\mathrm{i}}_3\right) = \lambda + \delta\lambda^0. \end{aligned} \tag{12}$$

Equation (12) is sometimes called a biquaternion. It should be noted that the designation “biquaternion” and “dual quaternion” are very relative, since they are equivalent and mean the same operator of the most general spatial transformation. Like the quaternion expression (5), the unit dual quaternion can be reduced to a trigonometric form:

$$\boldsymbol{\Lambda} = \cos\Phi + \mathbf{E}\sin\Phi.$$

Where $\mathbf{E}$ is the single screw of the screw part of the dual quaternion; is the dual argument (dual angle) of the dual quaternion E. Based on the Kotelnikov-Study principle of transfer, all formulas written for quaternions are unexpanded formulas for dual quaternions. For example, applying this principle to the rotation operation (6), we can write

$$\mathbf{R}' = \boldsymbol{\Lambda} \circ \mathbf{R} \ \circ \widetilde{\boldsymbol{\Lambda}}$$

the essence of which is expressed in the following - the screw $\mathbf{R}'$ is obtained by moving the screw $\mathbf{R}$ along the unit screw $\mathbf{E}$ to the dual angle 2Φ.

3. The construction of closed-loop equations for spatial mechanisms

It is well known that the compilation of the equations for the closure of spatial mechanisms is a laborious task. The derivation of the equations of interrelation between the parameters of the mechanism by performing multiplication operations in the closure equations is an almost impossible task. Traditional operators for spatial transformation by using matrix approach for complex spatial mechanisms gives the difficult nonlinear equations. The closed-loop equations for the spatial 7- link mechanism (Fig.1) are expressed by the dual quaternion product:

$$\boldsymbol{\Lambda}_1 \circ \mathbf{A}_1 \circ \boldsymbol{\Lambda}_2 \circ \mathbf{A}_2 \circ \cdots \circ \boldsymbol{\Lambda}_7 \circ \mathbf{A}_7 = 1, \tag{13}$$

where $\boldsymbol{\Lambda}_k = \cos\Phi_k + \bar{\mathrm{i}}_3 \sin\Phi_k \quad (k = 1, 2, \ldots, 7)$ the dual quaternions characterizing the displacements in kinematic pairs can be called "variables":$\Phi_k = \varphi_k + \delta\varphi_k^0$, and these angles (see Fig.1) are equal to half the angles Φ'_K: $\Psi_k = \frac{1}{2}\ \Phi'_K = \frac{1}{2}(\varphi'_k + \delta\varphi_k^{'0})$; $\mathbf{A}_k = \cos B_k + \bar{\mathrm{i}}_2 \sin B_k$, dual quaternions characterizing the dimensions of the links of the mechanism can be called "constant": $B_k = \beta_k + \delta_k^0$, these angles are also equal to half the angles B'_k: $B_k = \frac{1}{2}B'_k = \frac{1}{2}(\beta'_k + \delta\beta_k^{0'})$, $k = 1 \div 7$ (shown in Fig.1 only for the 1st link).

It was shown in [3] that equation (13) is the starting point for all single-loop mechanisms (including plane four-link mechanisms), where conditions for simplifying the closed-loop equations are also given.

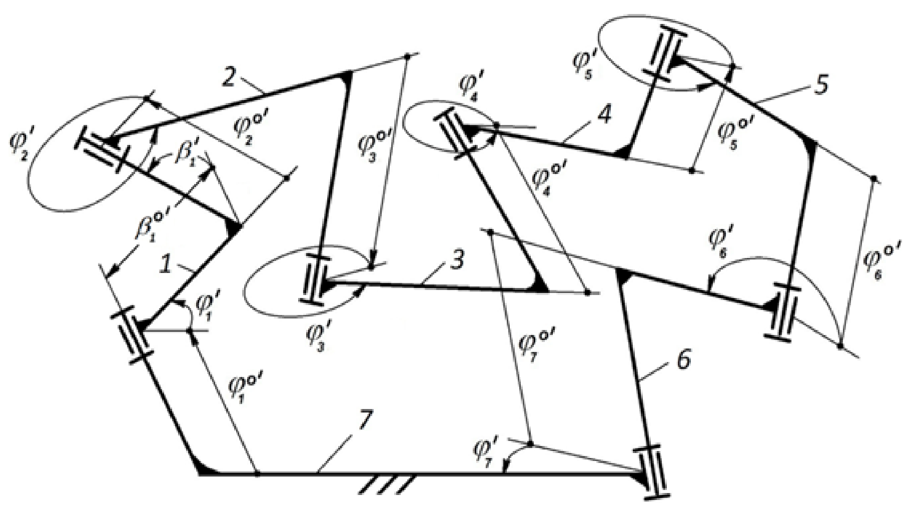

Figure 2. Spatial $7R$ mechanism.

4. Simplification of dual quaternion multiplication

To simplify the dual quaternion multiplication (13), the dual quaternion multipliers should be distributed on both sides of the equation:

$$\mathbf{\Lambda_1} \circ \mathbf{A_1} \circ \mathbf{\Lambda_2} \circ \mathbf{A_2} \circ \mathbf{\Lambda_3} \circ \mathbf{A_3} \circ \mathbf{\Lambda_4} = \widetilde{\mathbf{A}}_\mathbf{7} \circ \widetilde{\mathbf{\Lambda}}_\mathbf{7} \circ \widetilde{\mathbf{A}}_\mathbf{6} \circ \widetilde{\mathbf{\Lambda}}_\mathbf{6} \circ \widetilde{\mathbf{A}}_\mathbf{5} \circ \widetilde{\mathbf{\Lambda}}_\mathbf{5} \circ \widetilde{\mathbf{A}}_\mathbf{4}, \tag{14}$$

where $\widetilde{\mathbf{A}}_\mathbf{7}, \widetilde{\mathbf{\Lambda}}_\mathbf{7}, \widetilde{\mathbf{A}}_\mathbf{6}, \widetilde{\mathbf{\Lambda}}_\mathbf{6}, \widetilde{\mathbf{A}}_\mathbf{5}, \widetilde{\mathbf{\Lambda}}_\mathbf{5}, \widetilde{\mathbf{A}}_\mathbf{4}$ conjugate dual quaternions. Express dual quaternions in trigonometric form:

$$(\cos\Phi_1 + \bar{\mathfrak{i}}_3 \sin\Phi_1) \circ (\cos B_1 + \bar{\mathfrak{i}}_2 \sin B_1) \circ \cdots \circ (\cos B_3 + \bar{\mathfrak{i}}_2 \sin B_3) \circ (\cos\Phi_4 + \bar{\mathfrak{i}}_3 \sin\Phi_4) =$$
$$\cdots \circ (\cos B_7 + \bar{\mathfrak{i}}_2 \sin B_7) \circ (\cos\Phi_7 + \bar{\mathfrak{i}}_3 \sin\Phi_7) \circ \cdots \circ (\cos\Phi_4 + \bar{\mathfrak{i}}_3 \sin\Phi_4) \circ (\cos B_4 + \bar{\mathfrak{i}}_2 \sin B_4).$$

By doing the multiplication left and right sides of this dual quaternion expression, and grouping the terms with units $1, \mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3$ we obtain four dual equations. There is one dual relationship between the four equations for the norm of the dual quaternion. Thus, in the four dual equations, only three are independent. Therefore, rejecting one of the four equations, we obtain the following three independent dual equations (for compactness, the sine and cosine are denoted by S and C):

$$\begin{aligned}
&C\Phi_1 CB_1 CB_2 CB_3 C(\Phi_2+\Phi_3+\Phi_4) - S\Phi_1 CB_1 CB_2 CB_3 S(\Phi_2+\Phi_3+\Phi_4) - \\
&-C\Phi_1 CB_1 SB_2 SB_3 C(\Phi_2-\Phi_3+\Phi_4) + S\Phi_1 CB_1 SB_2 SB_3 S(\Phi_2-\Phi_3+\Phi_4) - \\
&-C\Phi_1 SB_1 CB_2 SB_3 C(\Phi_2+\Phi_3-\Phi_4) - S\Phi_1 SB_1 CB_2 SB_3 S(\Phi_2+\Phi_3-\Phi_4) - \\
&-C\Phi_1 SB_1 SB_2 CB_3 C(\Phi_2-\Phi_3-\Phi_4) - S\Phi_1 SB_1 SB_2 CB_3 S(\Phi_2+\Phi_3-\Phi_4) + \\
&+[SB_7 CB_6 CB_5 SB_4 - CB_7 CB_6 CB_5 CB_4] C(\Phi_5+\Phi_6+\Phi_7) + \\
&+[CB_7 SB_6 SB_5 CB_4 - SB_7 SB_6 SB_5 SB_4] C(\Phi_5-\Phi_6+\Phi_7) + \\
&+[CB_7 SB_6 CB_5 SB_4 + SB_7 SB_6 CB_5 CB_4] C(\Phi_5+\Phi_6-\Phi_7) + \\
&+[SB_7 CB_6 SB_5 CB_4 + CB_7 CB_6 SB_5 SB_4] C(\Phi_5-\Phi_6-\Phi_7) = 0, \\
&S\Phi_1 SB_1 CB_2 CB_3 C(\Phi_2+\Phi_3+\Phi_4) - C\Phi_1 SB_1 CB_2 CB_3 S(\Phi_2+\Phi_3+\Phi_4) - \\
&-S\Phi_1 SB_1 SB_2 SB_3 C(\Phi_2-\Phi_3+\Phi_4) + C\Phi_1 SB_1 SB_2 SB_3 S(\Phi_2-\Phi_3+\Phi_4) + \\
&+S\Phi_1 CB_1 CB_2 SB_3 C(\Phi_2+\Phi_3-\Phi_4) + C\Phi_1 CB_1 CB_2 SB_3 S(\Phi_2+\Phi_3-\Phi_4) + \\
&-S\Phi_1 CB_1 SB_2 CB_3 C(\Phi_2-\Phi_3-\Phi_4) + C\Phi_1 CB_1 SB_2 CB_3 S(\Phi_2-\Phi_3-\Phi_4) + \\
&+[SB_7 CB_6 CB_5 CB_4 - CB_7 CB_6 CB_5 SB_4] S(\Phi_5+\Phi_6+\Phi_7) + \\
&+[CB_7 SB_6 SB_5 SB_4 - SB_7 SB_6 SB_5 CB_4] S(\Phi_5-\Phi_6+\Phi_7) + \\
&+[CB_7 SB_6 CB_5 CB_4 - SB_7 SB_6 CB_5 SB_4] S(\Phi_5+\Phi_6-\Phi_7) + \\
&+[CB_7 CB_6 SB_5 CB_4 + SB_7 CB_6 SB_5 SB_4] S(\Phi_5-\Phi_6-\Phi_7) = 0, \\
&C\Phi_1 SB_1 CB_2 CB_3 C(\Phi_2+\Phi_3+\Phi_4) + S\Phi_1 SB_1 CB_2 CB_3 S(\Phi_2+\Phi_3+\Phi_4) - \\
&-C\Phi_1 SB_1 SB_2 SB_3 C(\Phi_2-\Phi_3+\Phi_4) - S\Phi_1 SB_1 SB_2 SB_3 S(\Phi_2-\Phi_3+\Phi_4) + \\
&+C\Phi_1 CB_1 CB_2 SB_3 C(\Phi_2+\Phi_3-\Phi_4) - S\Phi_1 CB_1 CB_2 SB_3 S(\Phi_2+\Phi_3-\Phi_4) + \\
&+C\Phi_1 CB_1 SB_2 CB_3 C(\Phi_2-\Phi_3-\Phi_4) - S\Phi_1 CB_1 SB_2 CB_3 S(\Phi_2-\Phi_3-\Phi_4) + \\
&+[-SB_7 CB_6 CB_5 CB_4 - CB_7 CB_6 CB_5 SB_4] C(\Phi_5+\Phi_6+\Phi_7) + \\
&+[\ SB_7 SB_6 SB_5 CB_4 + CB_7 SB_6 SB_5 SB_4] C(\Phi_5-\Phi_6+\Phi_7) + \\
&+[\ SB_7 SB_6 CB_5 SB_4 - CB_7 SB_6 CB_5 CB_4] C(\Phi_5+\Phi_6-\Phi_7) + \\
&+[\ SB_7 CB_6 SB_5 SB_4 - CB_7 CB_6 SB_5 CB_4] C(\Phi_5-\Phi_6-\Phi_7) = 0.
\end{aligned} \tag{15}$$

We introduce intermediate angles:

$$\begin{gathered}\Psi_1=\Phi_2+\Phi_3+\ \Phi_4;\Psi_2=\Phi_2-\Phi_3+\ \Phi_4;\Psi_3=\Phi_2+\Phi_3-\ \Phi_4;\Psi_4=\Phi_2-\Phi_3-\ \Phi_4,\\ \Psi_5=\Phi_5+\Phi_6+\Phi_7;\Psi_6=\Phi_5-\Phi_6+\Phi_7;\Psi_7=\Phi_5+\Phi_6-\Phi_7;\Psi_8=\Phi_5-\Phi_6-\Phi_7.\end{gathered}\tag{16}$$

We introduce the following notation:

$$\begin{aligned}
C_{1,1}=\ C\Phi_1CB_1CB_2CB_3;\quad C_{1,2}=-S\Phi_1CB_1CB_2CB_3;\\
C_{1,3}=-C\Phi_1CB_1SB_2SB_3;\quad C_{1,4}\ =S\Phi_1CB_1SB_2SB_3;\\
C_{1,5}=-C\Phi_1SB_1CB_2SB_3;\quad C_{1,6}=-S\Phi_1SB_1CB_2SB_3;\\
C_{1,7}=-C\Phi_1SB_1SB_2CB_3;\quad C_{1,8}=-S\Phi_1SB_1SB_2CB_3;\\
C_{1,9}=\ SB_7CB_6CB_5SB_4-CB_7CB_6CB_5CB_4;\quad C_{1,10}=0;\\
C_{1,11}=CB_7SB_6SB_5CB_4-SB_7SB_6SB_5SB_4;\quad C_{1,12}=0;\\
C_{1,13}=CB_7SB_6CB_5SB_4+SB_7SB_6CB_5CB_4;\quad C_{1,14}=0;\\
C_{1,15}=SB_7CB_6SB_5CB_4+CB_7CB_6SB_5SB_4;\quad C_{1,14}=0;\\
C_{2,1}=S\Phi_1SB_1CB_2CB_3;\quad C_{2,2}=-C\Phi_1SB_1CB_2CB_3;\\
C_{2,3}=-S\Phi_1SB_1SB_2SB_3;\quad C_{2,4}=C\Phi_1SB_1SB_2SB_3;\\
C_{2,5}=S\Phi_1CB_1CB_2SB_3;\quad C_{2,6}=C\Phi_1CB_1CB_2SB_3;\\
C_{2,7}=-S\Phi_1CB_1SB_2CB_3;\quad C_{2,8}=C\Phi_1CB_1SB_2CB_3;\\
C_{2,8}=0;\quad {}_{2,10}=SB_7CB_6CB_5CB_4-CB_7CB_6CB_5SB_4;\\
C_{2,11}=0;\quad {}_{2,12}=CB_7SB_6SB_5SB_4-SB_7SB_6SB_5CB_4;\\
C_{2,13}=0;\quad {}_{2,14}=CB_7SB_6CB_5CB_4-SB_7SB_6CB_5SB_4;\\
C_{2,15}=0;\quad {}_{2,16}=CB_7CB_6SB_5CB_4+SB_7CB_6SB_5SB_4;\\
C_{3,1}=C\Phi_1SB_1CB_2CB_3;\quad C_{3,2}=S\Phi_1SB_1CB_2CB_3;\\
C_{3,3}=-C\Phi_1SB_1SB_2SB_3;\quad C_{3,4}=-S\Phi_1SB_1SB_2SB_3;\\
C_{3,5}=C\Phi_1CB_1CB_2SB_3;\quad C_{3,6}=-S\Phi_1CB_1CB_2SB_3;\\
C_{3,7}=C\Phi_1CB_1SB_2CB_3;\quad C_{3,8}=-S\Phi_1CB_1SB_2CB_3;\\
C_{3,9}=-SB_7CB_6CB_5CB_4-CB_7CB_6CB_5SB_4;\quad C_{3,10}=0;\\
C_{3,11}=SB_7SB_6SB_5CB_4+CB_7SB_6SB_5SB_4;\quad C_{3,12}=0;\\
C_{3,13}=SB_7SB_6CB_5SB_4-CB_7SB_6CB_5CB_4;\quad C_{3,14}=0;\\
C_{3,15}=SB_7CB_6SB_5SB_4-CB_7CB_6SB_5CB_4;\quad C_{3,16}=0.
\end{aligned}$$

Thus, dual equations (15) briefly can be written in the following form:

$$\begin{gathered}C_{k,1}C\Psi_1+C_{k,2}S\Psi_1+C_{k,3}C\Psi_2+C_{k,4}S\Psi_2+C_{k,5}C\Psi_3+C_{k,6}S\Psi_3+C_{k,7}\ C\Psi_4+\\ C_{k,8}S\Psi_4+C_{k,9}C\Psi_5+C_{k,10}S\Psi_5+C_{k,11}C\Psi_6+C_{k,12}S\Psi_6+C_{k,13}C\Psi_7+C_{k,14}S\Psi_7+\\ C_{k,15}C\Psi_8+C_{k,16}S\Psi_8=0,\quad k=1,2,3.\end{gathered}\tag{17}$$

Dividing these equations into the main and moment parts are obtained following six equations with real numbers. The first three equations repeat these equations, but replacing capital letters with small letters:

$$\begin{gathered}c_{k,1}C\psi_1+c_{k,2}S\psi_1+c_{k,3}C\psi_2+c_{k,4}S\psi_2+c_{k,5}C\psi_3+c_{k,6}S\psi_3+c_{k,7}C\psi_4+\\ +c_{k,8}S\psi_4+c_{k,9}C\psi_5+c_{k,10}S\psi_5+c_{k,11}C\psi_6+c_{k,12}S\psi_6+c_{k,13}C\psi_7+c_{k,14}\times\\ \times S\psi_7+c_{k,15}C\psi_8+c_{k,16}S\psi_8=0,\quad k=1,2,3.\end{gathered}\tag{18}$$

In equations (18), the coefficients $c_{k,j}, (k = 1, 2, 3; j = 1, 2, ..., 16)$ are the main parts of the corresponding coefficients, for example:

$$c_{1,1} = \ C\varphi_1 C\beta_1 C\beta_2 C\beta_3.$$

The following three equations are the moment part of equations (18), [the sign " $\circ$ " means the moment part]:

$$\begin{gathered}(c^0_{k,1} + c_{k,2}\psi^0_1)C\psi_1 + (c^0_{k,2} - c_{k,1}\psi^0_1)S\psi_1 + (c^0_{k,3} + c_{k,4}\psi^0_2)C\psi_2 + (c^0_{k,4} - c_{k,3}\psi^0_2)S\psi_2+\\ (c^0_{k,5} + c_{k,6}\psi^0_3)C\psi_3 + (c^0_{k,6} - c_{k,5}\psi^0_3)S\psi_3 + (c^0_{k,7} + c_{k,8}\psi^0_4)C\psi_4 + (c^0_{k,8} - c_{k,7}\psi^0_4)S\psi_4+\\ (c^0_{k,9} + c_{k,10}\psi^0_5)C\psi_5 + (c^0_{k,10} - c_{k,9}\psi^0_5)S\psi_5 + (c^0_{k,11} + c_{k,12}\psi^0_6)C\psi_6 + (c^0_{k,12} - c_{k,11}\psi^0_6)S\psi_6+\\ (c^0_{k,13} + c_{k,14}\psi^0_7)C\psi_7 + (c^0_{k,14} - c_{k,13}\psi^0_7)S\psi_7+\\ (c^0_{k,15} + c_{k,16}\psi^0_8)C\psi_8 + (c^0_{k,16} - c_{k,15}\psi^0_8)S\psi_8 = 0.\end{gathered} \tag{19}$$

In these equations, the coefficients $c^0_{k,j}, \ \ (k = 1, 2, 3; j = 1, 2, ..., 16)$ are the moment parts of the corresponding coefficients, for example:

$$\begin{gathered}c^0_{1,1} = (C_{1,1})^0 = (C\Phi_1 CB_1 CB_2 CB_3)^0 = (C\Phi_1)^0 C\beta_1 C\beta_2 C\beta_3+\\ C\varphi_1(CB_1)^0 C\beta_2 C\beta_3 + \ C\varphi_1 C\beta_1(CB_2)^0 C\beta_3 + \ C\varphi_1 C\beta_1 C\beta_2(CB_3)^0 =\\ -\varphi^0_1 S\varphi_1 C\beta_1 C\beta_2 C\beta_3 - C\varphi_1\beta^0_1 S\beta_1 C\beta_2 C\beta_3 - C\varphi_1 C\beta_1\beta^0_2 S\beta_2 C\beta_3 - C\varphi_1 C\beta_1 C\beta_2\beta^0_3 S\beta_3.\end{gathered}$$

$\psi^0_1, (i = 1, 2, ..., 8)$ are the moment parts of the intermediate dual angles Ψ_i and are known quantities.

We introduce the notation:

$$\begin{gathered}C\psi_1 = x_1; S\psi_1 = x_2; C\psi_2 = x_3; S\psi_2 = x_4;\\ C\psi_3 = x_5; S\psi_3 = x_6; C\psi_4 = x_7; S\psi_4 = x_8;\\ C\psi_5 = x_9; S\psi_5 = x_{10}; \ C\psi_6 = x_{11}; \ S\psi_6 = x_{12};\\ C\psi_7 = x_{13}; S\psi_7 = x_{14}; C\psi_8 = x_{15}; S\psi_8 = x_{16}.\end{gathered} \tag{20}$$

With these notations, equations (18) and (19) take the following form:

$$\begin{gathered}c_{k,1}x_1 + c_{k,2}x_2 + c_{k,3}x_3 + c_{k,4}x_4 + c_{k,5}x_5 + c_{k,6}x_6 + c_{k,7}x_7 + c_{k,8}x_8+\\ c_{k,9}x_9 + c_{k,10}x_{10} + c_{k,11}x_{11} + c_{k,12}x_{12} + c_{k,13}x_{13} + c_{k,14}x_{14} + c_{k,15}x_{15} + c_{k,16}x_{16} = 0,\\ (c^0_{k,1} + c_{k,2}\psi^0_1)x_1 + (c^0_{k,2} - c_{k,1}\psi^0_1)x_2 + (c^0_{k,3} + c_{k,4}\psi^0_2)x_3 + (c^0_{k,4} - c_{k,3}\psi^0_2)x_4+\\ (^0_{k,5} + c_{k,6}\psi^0_3)x_5 + (^0_{k,6} - c_{k,5}\psi^0_3)x_6 + (c^0_{k,7} + c_{k,8}\psi^0_4)x_7 + (c^0_{k,8} - c_{k,7}\psi^0_4)x_8+\\ (c^0_{k,9} + c_{k,10}\psi^0_5)x_9 + (c^0_{k,10} - c_{k,9}\psi^0_5)x_{10} + (c^0_{k,11} + c_{k,12}\psi^0_6)x_{11} + (c^0_{k,12} - c_{k,11}\psi^0_6)x_{12}+\\ (c^0_{k,13} + c_{k,14}\psi^0_7)x_{13} + (c^0_{k,14} - c_{k,13}\psi^0_7)x_{14}+\\ (c^0_{k,15} + c_{k,16}\psi^0_8)x_{15} + (c^0_{k,16} - c_{k,15}\psi^0_8)x_{16} = 0.\\ k = 1, 2, 3.\end{gathered} \tag{21}$$

The following 8 equations reflect the obvious conditions:

$$\begin{gathered}x^2_1 + x^2_2 = 1; x^2_3 + x^2_4 = 1; x^2_5 + x^2_6 = 1; x^2_7 + x^2_8 = 1,\\ x^2_9 + x^2_{10} = 1; x^2_{11} + x^2_{12} = 1; x^2_{13} + x^2_{14} = 1.x^2_{15} + x^2_{16} = 1.\end{gathered} \tag{22}$$

Between intermediate angles, there are the following dependencies:

$$\begin{gathered}\psi_1 + \psi_4 = \psi_2 + \psi_3, \ \psi_5 + \psi_8 = \psi_6 + \psi_7 \ \text{ or}\\ \cos(\psi_1 + \psi_4) = \cos(\psi_2 + \psi_3), \ \cos(\psi_5 + \psi_8) = \cos(\psi_6 + \psi_7).\end{gathered}$$

Having revealed the last expressions, we get:

$$S\psi_1 C\psi_4 + C\psi_1 S\psi_4 = S\psi_2 C\psi_3 + C\psi_2 S\psi_3, S\psi_5 C\psi_8 + C\psi_5 S\psi_8 = S\psi_6 C\psi_7 + C\psi_6 S\psi_7,$$

or taking into account expressions (20):

$$\begin{aligned} &x_2 x_7 + x_1 x_8 = x_4 x_5 + x_3 x_6, \\ &x_{10} x_{15} + x_9 x_{16} = x_{12} x_{13} + x_{11} x_{14}. \end{aligned} \tag{23}$$

Thus, to determine the unknowns x_k, $(k = 1, 2, ..., 16)$ we have 16 equations - 6 equations from expression (21), 8 equations from expression (22), and two equations from expression (23). Having determined the intermediate angles, we easily determine the angles $\varphi_2, \varphi_3,\ \varphi_4, \varphi_5, \varphi_6, \varphi_7$.

Thus, as a result of the proposed method, the displacement analysis of the $7R$ mechanism in comparison with other methods is described by relatively simple equations and therefore, their numerical solution is not difficult.

Remark 4.1. The introductions of intermediate angles are also possible if the dual quaternions terms leave in one side of the closure equations, as in equation (13). By expanding the quaternion product (13), and equating the coefficients at orths $1, \mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3$ by the above method, we also obtain six independent equations. Accordingly, intermediate unknown angles will be as follows:

$$\Psi_k = \Phi_2 \pm \Phi_3 \pm \Phi_4 \pm \Phi_5 \pm \Phi_6 \pm \Phi_7, \qquad k = 1, 2, \ldots, 32.$$

These six equations will be linear with respect to the sine and cosines of the intermediate angles. Thus, the number of unknowns will reach 64. To determine the unknowns, in addition to the above six equations, there will be another 58 dependencies - 32 equations like (22) and 26 equations like (23). It is clear that this method is unprofitable from a mathematical point of view, and therefore quaternions-factors should be uniformly distributed on both sides of the equation of the condition of closure of the mechanism.

5. A numerical example

As a numerical example, displacement problem of the $7R$ mechanism with the following design dimensions was solved:

$$\begin{aligned} &\beta_1' = 10^0, \beta_2' = 15^0, \beta_3' = 15^0, \beta_4' = 20^0, \beta_5' = 15^0, \beta_6' = -15^0, \beta_7' = -10^0, \\ &\beta_1^{0\prime} = 3, \beta_2^{0\prime} = 4, \beta_3^{0\prime} = 5, \beta_4^{0\prime} = 2, \beta_5^{0\prime} = 2, \beta_6^{0\prime} = 3, \beta_7^{0\prime} = 3, \\ &\varphi_1^{0\prime} = 1, \varphi_2^{0\prime} = 1, \varphi_3^{0\prime} = 2, \varphi_4^{0\prime} = 2, \varphi_5^{0\prime} = 2, \varphi_6^{0\prime} = 3, \varphi_7^{0\prime} = 3. \end{aligned}$$

Calculations are made for the discrete input positions of the first link through every 15 degrees of its turn. For each position of the input link, the angles $\varphi_2,\ \varphi_3,\ \varphi_4,\ \varphi_5, \varphi_6,\ \varphi_7$ are calculated. Iterative process produced by the method of Newton-Raphson. The following values are displayed on the screen: f1-angle of rotation of the input link, it is the quantity of iterations for the given step, $f2, f3, f4, f5, f6, f7$ the corresponding angles rotations in the rotational kinematic pairs. The results of the numerical example showed that, during the rotation of the input link, the mechanism go over from one assembly to another. As can be seen from the numerical example, beginning with $\varphi_1 = 345^0$ (indicated in bold letters), the mechanism is transferred to a stable assembly. Therefore, to obtain a stable assembly, the input link is informed of two turns.

Table 2. Results of the numerical solutions.

$\varphi 1$	it.	$\varphi 2$	$\varphi 3$	$\varphi 4$	$\varphi 5$	$\varphi 6$	$\varphi 7$
0	12	229,8604	119,9979	49,9596	163,6792	153,7389	163,6971
15	12	267,3578	156,7464	1,2869	142,5680	137,5505	163,7591
30	5	282,7395	144,4137	1,3539	106,3417	150,7853	172,3097
45	5	280,2083	280,2083	5,8589	93,5090	160,3827	176,1918
60	4	273,7832	98,0836	11,6395	86,4454	168,3609	178,0511
75	4	264,1061	64,1502	20,7367	82,3210	172,4578	177,9998
90	7	217,5709	297,0543	64,2346	112,9116	179,8616	154,8706
105	5	188,8891	237,6696	75,8594	135,2061	205,6596	141,2783
120	4	184,3510	214,3974	78,8672	136,7380	205,9260	140,2375
135	4	182,8586	196,1283	81,0316	135,9858	204,5073	141,4451
150	4	181,9645	180,3209	82,2831	134,8478	203,8835	143,8176
165	4	181,1723	166,4852	82,6935	133,6272	204,3064	146,8744
180	3	180,3906	154,3912	82,5044	132,3906	205,7279	150,2851
195	3	179,6045	143,7799	81,9730	131,1865	208,1048	153,8053
210	3	178,7874	134,3346	81,2926	130,0499	211,3845	157,2599
225	3	177,8822	125,7024	80,5741	128,9798	215,4501	160,5341
240	3	176,7982	117,5104	79,8542	127,9283	220,0882	163,5633
255	3	175,4007	109,3587	79,1066	126,7961	224,9720	166,3217
270	3	173,4783	100,7775	78,2431	125,4164	229,6272	168,8099
285	3	170,6476	91,1180	77,0777	123,4899	233,3113	171,0500
300	3	166,0474	79,2581	75,1476	120,3435	234,5753	173,1062
315	3	157,0457	62,6383	70,7299	113,8314	229,3464	175,2396
330	4	133,5780	35,8906	54,4456	95,6297	203,2686	178,8825
345	5	90,6280	1,7108	19,8272	73,3145	166,3865	184,3940
360	5	68,0151	48,0000	1,0410	86,6697	195,7284	188,5856
375	5	25,1047	96,1466	341,6684	106,6827	216,1925	188,1032
390	5	329,9114	117,6188	341,8062	107,3038	201,6085	183,2379
405	4	300,7327	109,3409	354,5212	96,3334	182,1828	181,7112
420	5	279,1345	93,5333	7,9831	86,8005	165,2766	179,9767
435	5	259,9527	72,2531	24,3111	83,4349	155,5610	176,4309
450	4	243,3743	48,2677	43,4438	89,2298	158,8294	170,8575
465	4	230,7729	26,5980	60,4914	99,6613	170,7876	165,4086
480	4	221,3460	8,6921	72,9136	109,6864	183,7948	161,4961
495	4	213,7265	354,3594	80,8766	117,7033	194,8670	159,0670
510	4	207,0308	343,1485	85,1332	123,4423	203,2716	157,7636
525	4	200,7503	334,4829	86,5194	127,0572	209,0319	157,2714
540	4	194,5913	327,7562	85,8196	128,8326	212,4208	157,3695
555	4	188,3746	322,4159	83,7076	129,0589	213,7564	157,9147
570	4	181,9749	318,0093	80,7277	127,9786	213,3162	158,8161
585	4	175,2812	314,2000	77,3000	125,7687	211,3069	160,0137
600	4	168,1704	310,7775	73,7292	122,5381	207,8562	161,4678
615	4	160,4850	307,6798	70,2044	118,3281	203,0123	163,1572
630	4	152,0140	305,0548	66,7697	113,1042	196,7411	165,0885
645	4	142,4804	303,4033	63,2314	106,7308	188,9223	167,3237
660	4	131,5712	303,9058	58,9272	98,9200	179,3892	170,0373
675	4	119,1135	309,1970	52,2154	89,2323	168,3145	173,6072
690	4	105,5152	325,4377	39,6981	77,9884	158,8277	178,5842
705	5	90,6280	1,7108	19,8272	73,3145	166,3865	184,3940
720	5	68,0151	48,0000	1,0410	86,6697	195,7284	188,5856

Based on the results of the numerical solution of the $7R$ mechanism, the graphs of the dependences $\varphi_k = \varphi_k(\varphi_1)$, $k = 2 \div 7$ were constructed (Fig.3). As one would expect, the angle φ_2 varies according to a constantly decreasing law, and the remaining angles $\varphi_{3\div 7}$ vary according to the periodic law having the maximum and minimum values.

6. Conclusion

We devised a new method for closed-loop equations of mechanisms that is particularly effective in the preparation of these equations for complex multi-loop spatial mechanisms. The offered method greatly simplified outline of the closed-loop equations of spatial mechanisms, whereby it becomes possible to express these equations in an explicit form. The numerical solution of the spatial $7R$ mechanism describe the decreasing and periodic laws of output six angles as functions of one input angle.

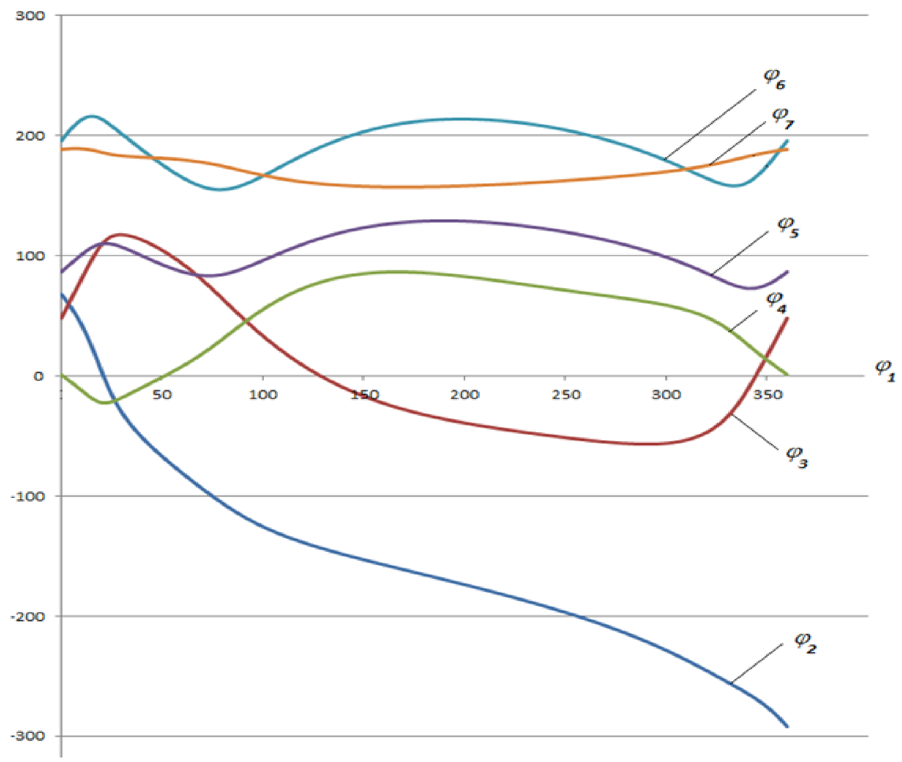

Figure 3. The graphs of the dependences $\varphi_k = \varphi_k(\varphi_1)$ ($k = 2 \div 7$).

Highlights.

(1) the displacement analysis of $7R$ spatial mechanism have been solved by using dual quaternion mathematical approach;
(2) in the preparation of closed-loop equations, we introduced intermediate angles;
(3) as a result, it was possible greatly to simplify the closed-loop equations and obtain algebraic equations of displacement of the $7R$ mechanism in the final form;
(4) by numerical solution 16 closed-loop equations we determine intermediate angles and then are calculation the angles of rotation in the kinematic pairs;

References

[1] Branets, V.N., Shmyglevsky, I.P., (1973), Application of Quaternions in Problems of Orientation of a Rigid Body, M., Science.

[2] Chelnokov, Yu.N., (2019), Generalization of Hamilton-Ishlinskii solid angle theorem for spatial motion of a solid body and its applications, Mechanics of Solids, 54, pp.1227-123.

[3] Chen, W-R., (1987), A matrix method of displacement analysis of the general spatial $7R$ mechanism, Appl. Math. Mech., 8(10), pp.957-967, doi:10.1007/BF02454258.

[4] Dantam, N.T. (2021), Robust and efficient forward, differential, and inverse kinematics using dual quaternions, Int. J. Rob. Res., 40(10-11), pp.1087-1105.

[5] Denavit, J., Hartenberg, R.S., (1955), A kinematic notation for lower-pair mechanisms based on matrices, ASME J. Appl. Mech., 22(2), pp.215-221, doi:10.1115/1.4011045.

[6] Dimentberg, F.M., (1978), The Theory of Screws and its Applications, M., Science, 327p.

[7] Duffy, J., Crane, C., (1980), A displacement analysis of the general spatial 7-link, $7R$ mechanism, Mech. Mach. Theory., 15(3), pp.153-169.

[8] Duffy, J., Derby, S., (1979), Displacement analysis of a spatial $7R$ mechanism - A Generalized Lobster's Arm, J. Mech. Des., 101(2), pp.224-231, doi:10.1115/1.3454042.

[9] Duffy, J., (1977), Displacement analysis of spatial seven-link 5R-2P mechanisms, J. Manuf. Sci. Eng., 99(3), pp.692-701, doi:10.1115/1.3439300.

[10] Freudenstein, F., (1973), Kinematics: past, present and future, Mech. Mach. Theory, 8(2), pp.151-161.

[11] Gallardo-Alvarado, J., Rodriguez-Castro, R., Perez-Gonzalez, L., Aguilar-Najera, C., (2018), Kinematics of the 3(RPSP)-S fully spherical parallel manipulator by means of screw theory, Robotics 7(2):29, doi:10.3390/robotics7020029.

[12] Gan, D., Liao, Q., Wei, S., Dai, J.S., Qiao, S., (2008), Dual quaternion-based inverse kinematics of the general spatial $7R$ mechanism, Proceedings of the Institution of Mechanical Engineers, Part C: J. Mech. Eng. Sci., 222(8), pp.1593-1598, doi:10.1243/09544062JMES1082

[13] Lee, H.Y., Liang, Ch.G., (1988), Displacement analysis of the general spatial 7-link $7R$ mechanism, Mech. Mach. Theory, 23(3),pp.219-226.

[14] Li, J., (2019), Research on the rigid body pose estimation using dual quaternions, Adv. Mech. Eng., 11(1), pp.1-11, doi:10.1177/1687814018823115.

[15] Lin, P.F., Huang, M.B., Huang, H.P., (2019), Analytical solution for inverse kinematics using dual quaternions, Open Access Journal, 14, doi:10.1109/Access.2019.2953553.

[16] Mammadov, F.H., (1994), Algebra of biquaternions in problems of analysis and synthesis of mechanisms, Dissertation work for the degree of candidate of technical sciences.

[17] Mammadov, F.H., (2017), Particular drawing biquaternion closure equations of complex spatial mechanisms, Proceedings of the second International Symposium of Mechanism and Machine Science, Baku, pp.87-92.

[18] Pankov, P.S., Zheentaeva Zh.K., Shirinov, T., (2021), Asymptotic reduction of solution space dimension for dynamical systems, TWMS J. Pure Appl. Math., 12(2), pp.243-253.

[19] Pfurner, M., Kong, X., Huang, Ch., (2014), Complete kinematic analysis of single-loop multiple-mode 7-link mechanisms based on Bennett and over constrained RPRP mechanisms, Mech. Mach. Theory., 73, pp.117-129.

[20] Sandor, G.N., Xu, Y.X., Weng, T.Ch., (1986), Synthesis of 7-R spatial motion generators with prescribed crank rotations and elimination of branching, Int. J. Rob. Res., 5(2), pp.143-156, doi:10.1177/027836498600500215.

[21] Valverde, A., Tsiotras, P., (2018), Spacecraft robot kinematics using dual quaternions, Robotics, 7(4), 64, doi:10.3390/robotics7040064.

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